On Unbiased Estimation of Sparse Vectors Corrupted by Gaussian Noise

What is the best possible performance of unbiased estimators in the sparse setting?
What do we learn from this about the sparse estimation challenge in general?

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Signal Model

$$\mathbf{y} = \mathbf{x}_0 + \mathbf{w}$$

- $\mathbf{w}$: White Gaussian noise
- $\mathbf{x}_0$: S-sparse, deterministic unknown $\mathbf{x}_0 \in \mathbb{R}^N$, $\|\mathbf{x}_0\|_0 \leq S < N$

Summary

- Analytical characterization of the best possible MSE performance of unbiased estimators
- Results provide:
  - Understanding of high-SNR performance
  - Identification of threshold region

Unbiasedness

- Unbiased estimator: $E_{\mathbf{x}_0}\{\hat{\mathbf{x}}\} = \mathbf{x}_0$ for all $\mathbf{x}_0$ with $\|\mathbf{x}_0\|_0 \leq S$

The unbiasedness assumption is required for bounds on the MSE to be nontrivial.

- **Theorem:** In the AWGN model without sparsity constraints, there exists only one unbiased estimator, namely, $\hat{\mathbf{x}} = \mathbf{y}$.
  However, with sparsity constraints, there are infinitely many unbiased estimators.

There are many unbiased estimators, so bounds on their performance are of practical importance.

- **Theorem:** For the sparse signal model, a uniformly minimum variance unbiased (UMVU) estimator does not exist.
  In other words, no estimator uniformly minimizes the mean-squared error (MSE) among all unbiased estimators.

Rather than finding the single best estimator, we will characterize the achievable MSE for every $\mathbf{x}_0$.

Characterizing the Barankin Bound (BB)

- **Goal:** For each sparse $\mathbf{x}_0$, find the minimum MSE achievable by any unbiased estimator.
- This value is known as the **Barankin bound**,

$$\mathrm{BB}(\mathbf{x}_0) \triangleq \inf_{\hat{\mathbf{x}} \in \mathcal{U}} \mathrm{MSE}(\hat{\mathbf{x}}, \mathbf{x}_0), \qquad \text{where } \mathcal{U} = \{\text{all unbiased } \hat{\mathbf{x}}\}.$$

- Computing the BB is not tractable; instead, we will bound it from below and above.

Lower Bound ≤ Barankin Bound ≤ Upper Bound

Any lower bound on the MSE of unbiased estimators is also a lower bound on BB.

- **Theorem:** Cramér-Rao bound *(Ben-Haim and Eldar, 2009)*

$$\mathrm{MSE}(\hat{\mathbf{x}}, \mathbf{x}_0) \geq \begin{cases} S\sigma^2, & \|\mathbf{x}_0\|_0 = S \\ N\sigma^2, & \|\mathbf{x}_0\|_0 < S \end{cases}$$

- **Theorem:** Hammersley-Chapman-Robbins bound

$$\mathrm{MSE}(\hat{\mathbf{x}}, \mathbf{x}_0) \geq \begin{cases} S\sigma^2 + (N-S-1)\sigma^2 e^{-|x_{\min}|^2/\sigma^2}, & \|\mathbf{x}_0\|_0 = S \\ N\sigma^2, & \|\mathbf{x}_0\|_0 < S \end{cases}$$

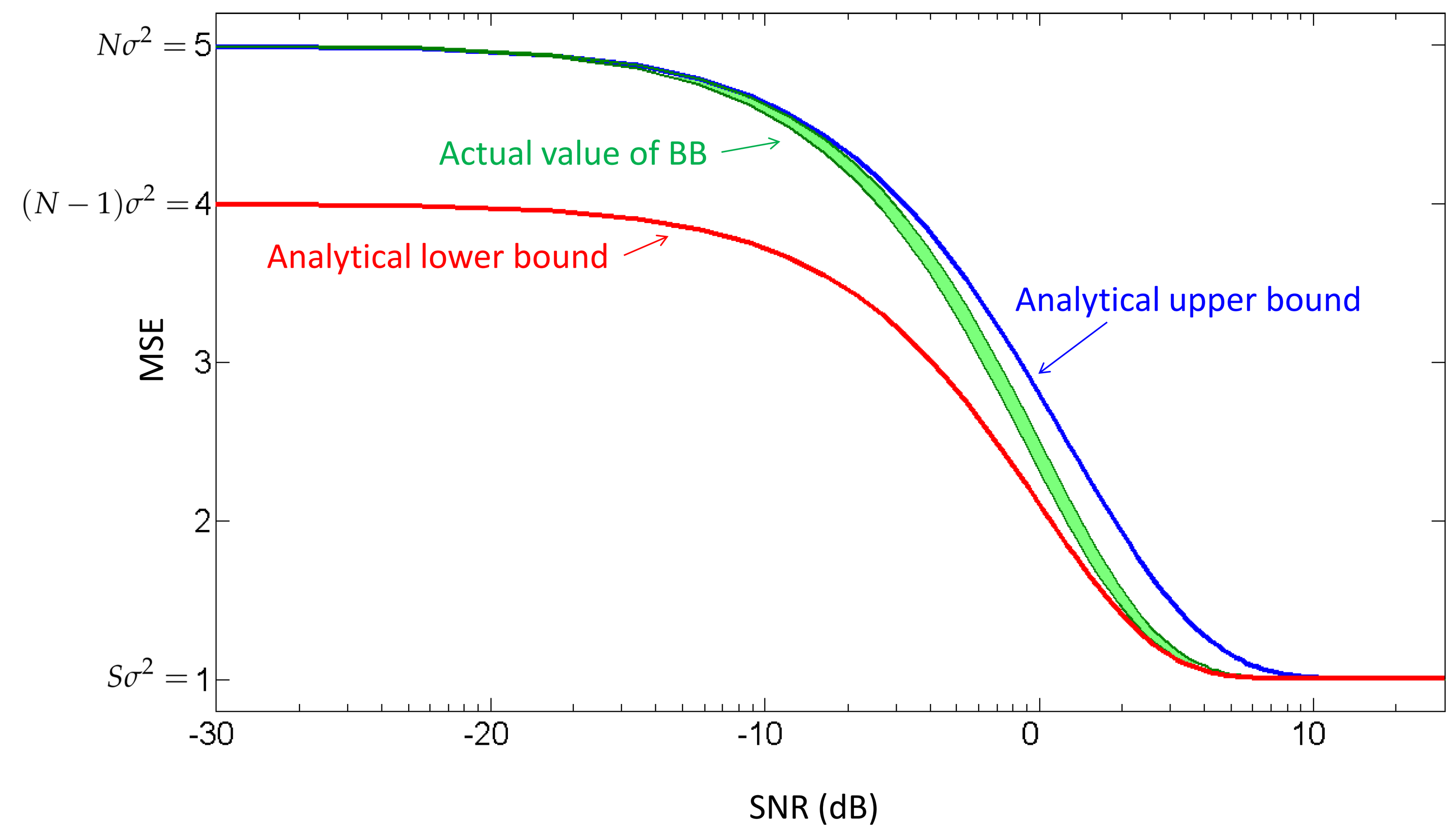
where $|x_{\min}|$ is the smallest (in magnitude) nonzero element in $\mathbf{x}_0$.

- **Theorem:** The Barankin bound is upper-bounded by

$$\mathrm{BB} \leq \begin{cases} S\sigma^2 + (N-S)\sigma^2 \Big(1 - \prod_{\ell \in \mathrm{supp}(\mathbf{x}_0)} g((\mathbf{x}_0)_\ell)\Big), & \|\mathbf{x}_0\|_0 = S \\ N\sigma^2, & \|\mathbf{x}_0\|_0 < S \end{cases}$$

where

$$g(x) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-\frac{1}{2\sigma^2}(x^2+y^2)} \sinh\left(\frac{xy}{\sigma^2}\right) \tanh\left(\frac{xy}{\sigma^2}\right) dy.$$



- Both lower and upper bounds can be improved numerically.
  This yields a more accurate (but numerical) characterization of the BB.

Conclusions

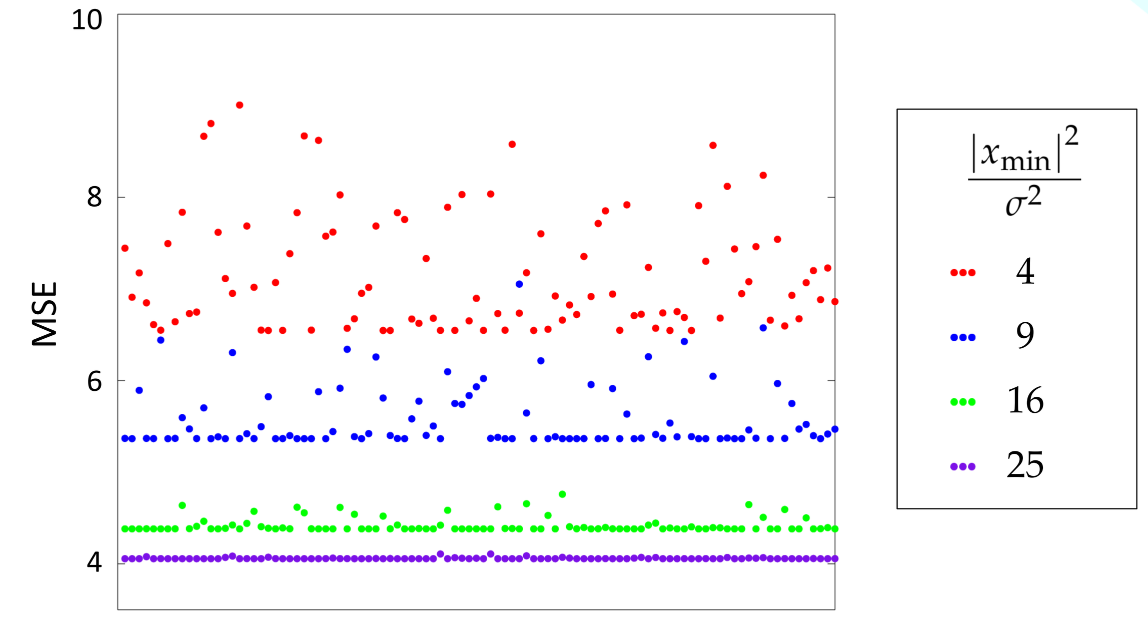
High-SNR Performance

- For high SNR,

$$s + C_1 \exp\left(-\frac{|x_{\min}|^2}{\sigma^2}\right) \leq \frac{\mathrm{BB}(\mathbf{x}_0)}{\sigma^2} \leq s + C_2 \exp\left(-\frac{|x_{\min}|^2}{2\sigma^2}\right)$$

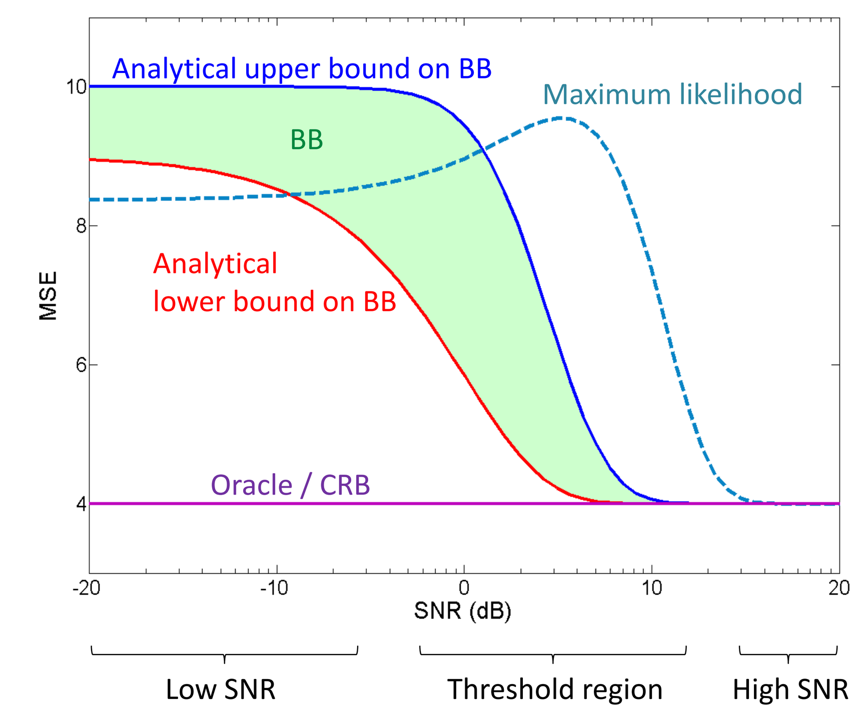
Smallest nonzero component $|x_{\min}|$ is the primary factor affecting performance

- This appears to be the case for biased estimators as well



Performance of the maximum likelihood estimator for various parameter values having a given value of $|x_{\min}|^2/\sigma^2$

Threshold Region Identification



- Practical estimators outperform BB at low SNR:
  In this case shrinkage is very effective, but is not allowed for unbiased estimators
- BB is similar to the performance of practical estimators at high SNR: Unbiased estimators are optimal in this case
- Threshold region is approximately indicated by BB