χ^2 and Noncentral χ^2 Distributions

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1 χ^2 Distribution

Definition 1. The χ^2 distribution is the sum of the squares of zero-mean Gaussian random variables. If $\{X_i\}_{i=1}^p$ are i.i.d. Gaussian random variables with zero mean and variance 1, then $Y = \sum_{i=1}^p X_i^2$ is distributed as χ^2 with k degrees of freedom (denoted χ_k^2).

The probability density function of Y is given by

$$f_Y(y) = \frac{y^{(p-2)/2}}{2^{p/2}\Gamma(p/2)} e^{-y/2}. \tag{1}$$

Basic properties of the χ^2 distribution are listed below [2, §6.3].

$$E(Y) = p \tag{2}$$

$$E\left(Y^2\right) = p(p+2) \tag{3}$$

$$E\left(\frac{1}{Y}\right) = \frac{1}{p-2}. \tag{4}$$

For $p > 2$ and $a > 0$, it can also be shown[1] that

$$E\left(\frac{1}{a+Y}\right) = \int_0^\infty f_Y(y)\frac{1}{a+y}dy$$

$$= \left(\frac{a}{2}\right)^{p/2} \frac{e^{a/2}}{a} \Gamma\left(1-\frac{p}{2}, \frac{a}{2}\right), \tag{5}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function.

Zero-mean Gaussian vectors which are not independent, or whose variance is not 1, can also be related to the χ^2 distribution, as follows. Let $\mathbf{X}$ be a zero-mean Gaussian p-vector with covariance $E(\mathbf{X}\mathbf{X}^*) = \mathbf{C}_\mathbf{X}$. Then, the random variable $\mathbf{X}^*\mathbf{C}_\mathbf{X}^{-1}\mathbf{X}$ is distributed as χ_p^2.

2 Noncentral χ^2 Distribution

Definition 2. The noncentral χ^2 distribution is the sum of the squares of non-zero-mean Gaussian random variables. Let $\{X_i\}_{i=1}^p$ be i.i.d. Gaussian random variables with means $\{\mu_i\}_{i=1}^p$, respectively, and variance 1. Then $Z = \sum_{i=1}^p X_i^2$ is distributed as noncentral χ^2 with p degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2}\boldsymbol{\mu}^*\boldsymbol{\mu}$. We will denote this as $\chi_p'^2(\lambda)$.

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[1] The integral was solved using Maple.

Note that in some references, the noncentrality parameter is defined as $\lambda = \mu^*\mu$, but we will not use this notation here.

The noncentral χ^2 distribution can be viewed as a χ^2 distribution with $p + 2K$ degrees of freedom, where K is a Poisson random variable with parameter λ. Thus, the probability distribution function is given by

$$f_Z(z) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} f_{Y_{p+2i}}(z), \tag{6}$$

where Y_q is distributed as χ^2_q.

Some additional properties of the noncentral χ^2 distribution are [2, §6.3], [3, p. 134]

$$E(Z) = p + 2\lambda \tag{7}$$

$$\mathrm{Var}(Z) = 2p + 8\lambda \tag{8}$$

$$E\left(\frac{1}{Z}\right) = E\left(\frac{1}{p + 2(K-1)}\right), \tag{9}$$

where $K \sim \mathrm{Poisson}(\lambda)$.

Using (5) it can be shown that, for $p > 2$ and $a > 0$,

$$E\left(\frac{1}{a+Z}\right) = \frac{e^{a/2}}{a} E\left[\left(\frac{a}{2}\right)^{(p+2K)/2} \Gamma\left(\frac{2-p-2K}{2}, \frac{a}{2}\right)\right], \tag{10}$$

where $K \sim \mathrm{Poisson}(\lambda)$ as before.

The inverse moments $E(1/Z^n)$, when $p > 2n$, were calculated by [1]. They are, for even p,

$$E\left(\frac{1}{Z^n}\right) = \frac{(-1)^{n-p/2}2^{-n}}{(n-1)!} \sum_{s=0}^{n-1} \left[\binom{n-1}{s} \lambda^{s-p/2+1} \Gamma\left(\frac{p}{2} - s - 1\right) \left(e^{-\lambda} - \sum_{t=0}^{p/2-s-2} \frac{(-\lambda)^t}{t!}\right)\right], \tag{11}$$

and for odd p,

$$E\left(\frac{1}{Z^n}\right) = \frac{(-1)^{n(p-1)/2}2^{-n}}{(n-1)!} \cdot \sum_{s=0}^{n-1} \left[\binom{n-1}{s} \lambda^{s-p/2+1} \Gamma\left(\frac{p}{2} - s - 1\right) \left(\frac{2}{\sqrt{\pi}} D\left(\sqrt{\lambda}\right) - \sqrt{\lambda} \sum_{m=0}^{\frac{p-5}{2}-s} \frac{(-\lambda)^m}{\Gamma\left(m + \frac{3}{2}\right)}\right)\right], \tag{12}$$

where Dawson's integral is given by

$$D(y) = e^{-y^2} \int_0^y e^{t^2} dt \quad \cong \frac{1}{2y} \text{ for large } y. \tag{13}$$

References

[1] M. E. Bock, G. G. Judge and T. A. Yancey (1984), "A simple form for the inverse moments of noncentral χ^2 and F random variables and certain confluent hypergeometric functions," Journal of Econometrics, **25**: 217–234.

[2] E. Greenberg and C. E. Webster, Jr. (1983), *Advanced Econometrics: A Bridge to the Literature,* Wiley.

[3] N. L. Johnson and S. Kotz (1970), *Continuous Univariate Distributions,* vol. 2, Houghton-Mifflin.